Fundamental Theorem of Calculus

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1 Statement of the Theorem

The fundamental Theorem of Calculus is one of the most important theorems in the history of mathematics, which was first discovered by Newton and Leibniz independently. This theorem reveals the underlying relation between differentiation and integration, which glues the two subjects into a uniform one, called calculus.

Theorem 1. *(The Fundamental Theorem of Calculus) Suppose $f(x)$ is a continuous function on $[a, b]$. Then:*

i). $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

ii). $\int_a^b f(x)\,dx = F(b) - F(a)$*, where F is an antiderivative of f*

Remark 2. We sometimes also write $F(b) - F(a)$ in the way $F(x)\Big|_a^b$, so Part (ii) of the theorem can also be written as $\int_a^b f(x)\,dx = F(x)\Big|_a^b$

Now we are going to look into each of these two parts of the theorem.

2 Part I

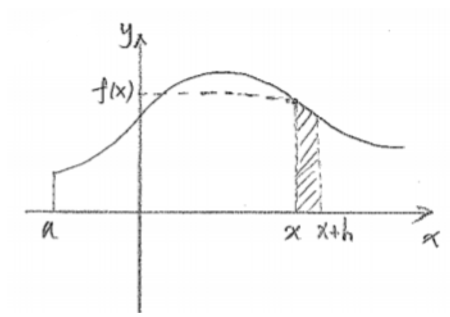
Let $g(x) = \int_a^x f(t)\,dt$, note $g(x)$ is a function of x. Part (i) tells us that $\frac{d}{dx}g(x) = f(x)$, i.e. $g(x)$ is an antiderivative of $f(x)$.

First we see a geometric explanation: $\int_0^x f(t)\,dt$ is the signed area bounded by $f(t)$ and the interval $[a, x]$. If we increase x to $x + h$, the increase in the area is $\int_x^{x+h} f(t)\,dt \approx f(x)h$ when h is a small number. This indicates

$$f(x) \approx \frac{\int_x^{x+h} f(t)\,dt}{h} \approx \frac{g(x+h)-g(x)}{h}$$

when h is small, so it implies

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$



Now let's formulate an algebraic proof.

Proof. Define $g(x)$ in the same way as above, we have

$$\frac{g(x+h)-g(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \frac{\int_x^{x+h} f(t)\,dt}{h}$$

We may assume $h > 0$, the other case can be discussed in a similar fashion.

Consider the continuous function f on the closed interval $[x, x+h]$. We know it has absolute maximum M and absolute minimum m on $[x, x+h]$, so $mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$.

Assume $f(u) = M$ and $f(v) = m$, where u, v are in $[x, x+h]$. We get:

$$f(v)h \leq \int_x^{x+h} f(t)\,dt \leq f(u)h$$

$$f(v) \leq \frac{\int_x^{x+h} f(t)\,dt}{h} \leq f(u)$$

$$f(v) \leq \frac{g(x+h)-g(x)}{h} \leq f(u)$$

If we take $h \to 0$, since u, v are in $[x, x+h]$, it follows $u \to x$ and $v \to x$, so $f(u) \to f(x)$ and $f(v) \to f(x)$, the last line of the inequalities implies

$$\lim_{h\to 0} f(v) \leq \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \leq \lim_{h\to 0} f(u)$$

i.e. $f(x) \leq g'(x) \leq f(x)$, we see $g'(x) = f(x)$. □

Example 3. *Find* $\frac{d}{dx}\int_1^{x^2} e^{3t}\,dt$

Let $g(x) = \int_1^x e^{3t}\,dt$, *we see* $\int_1^{x^2} e^{3t}\,dt = g(x^2)$. *The Fundamental Theorem of Calculus says* $g'(x) = e^{3x}$. *By the chain rule we get*

$$\frac{d}{dx}g(x^2) = g'(x^2)(x^2)' = e^{3(x^2)}(2x) = 2xe^{3x^2}$$

Example 4. *Compute* $\frac{d}{dx}\int_{2x+1}^{3x-5} t^2\,dt$

Observe that $\int_{2x+1}^{3x-5} t^2\,dt = \int_0^{3x-5} t^2\,dt - \int_0^{2x+1} t^2\,dt$, *we see*

$$\begin{aligned}\frac{d}{dx}\int_{2x+1}^{3x-5} t^2\,dt &= \frac{d}{dx}\int_0^{3x-5} t^2\,dt - \frac{d}{dx}\int_0^{2x+1} t^2\,dt \\ &= (3x-5)^2(3x-5)' - (2x+1)^2(2x+1)' \\ &= (3x-5)^2 \times 3 - (2x+1)^2 \times 2 \\ &= 3(3x-5)^2 - 2(2x+1)^2\end{aligned}$$

Definition 5. f is integrable on $[a, b]$, define the average of f on $[a, b]$ to be the number

$$\bar{f}_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 6. *If* $v(t)$ *is a velocity function, then the average pf* $v(t)$ *on* $[a, b]$, $\bar{v}_{[a,b]} = \frac{1}{b-a}\int_a^b v(t)\,dt$, *is defined to be the average velocity on* $[a, b]$.

3 Part II

We first give a proof.

Proof. By Part (i), $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, so $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$. If $F(x)$ is another antiderivative of $f(x)$, we know that $F(x) = g(x) + C$ for some constant C. This implies

$$F(b) - F(a) = g(b) - g(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt$$

□

Remark 7. There is an intuitive explanation of Part (ii) as well.

We know that $\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$, so when we take $\max \Delta x$ very small, we have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n f(x_i^*)\Delta x_i$$

Now on each $[x_{i-1}, x_i]$, since this is a small interval, we have

$$f(x^*) = F'(x^*) \approx \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_i) - F(x_{i-1})}{\Delta x_i}$$

so

$$F(x_i) - F(x_{i-1}) \approx f(x^*)\Delta x_i$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n f(x_i^*)\Delta x_i \approx \sum_{i=1}^n F(x_i) - F(x_{i-1}) = F(b) - F(a)$$

finally, when we take the limit $\max \Delta x_i \to 0$, we get the equality.

Example 8. *Evaluate* $\int_1^3 e^x\,dx$

An antiderivative of $f(x) = e^x$ *is* $F(x) = e^x$*, so by the Fundamental Theorem of Calculus,*

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

Definition 9. The indefinite integral of a function f is defined to be $\int f(x)\,dx = F(x) + C$, where $F'(x) = f(x)$, i.e. $F(x)$ is an antiderivative of $f(x)$, and C denotes a constant.

Example 10. *Evaluate* $\int_0^3 (x^3 - 4x)\,dx$

Since $\int (x^3 - 4x)\,dx = \frac{1}{4}x^4 - 2x^2 + C$, *we have*

$$\int_0^3 (x^3 - 4x)\,dx = (\frac{1}{4} \times 3^4 - 2 \times 3^2) - (\frac{1}{4} \times 0^4 - 2 \times 0^2) = \frac{9}{4}$$

4 Applications

Example 11. *We are going to prove the area formula for circles.*

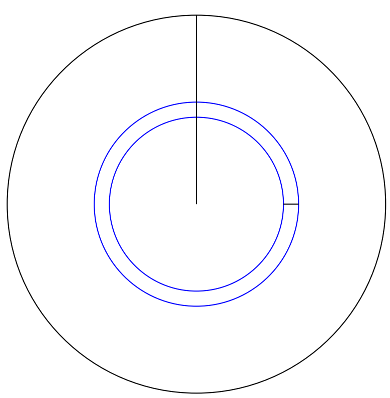
First, recall the definition of the number π*:* π *is the ratio of the circumference and diameter of a circle. By this definition, we know that the circumference of a circle of radius* R *is* $2\pi R$*, since* $2R$ *is the diameter.*

Now given a circle of radius R*, we are going to find its area. We divide* $[0, R]$ *into* n *subintervals of equal length* $\Delta r = \frac{R}{n}$*, with endpoints* $r_0 = 0, r_1, ..., r_{n-1}, r_n = R$*, and by the following picture, we see that when* n *is getting big, the are of the circle can be approximated by the following:*

$$R_n = \sum_{i=1}^{n} (2\pi r_i)\Delta r$$

Taking the limit, we get the area of the circle is

$$\lim_{n\to\infty} R_n = \int_0^R 2\pi r\,dr = \pi r^2 \Big|_0^R = \pi R^2$$



Example 12. *We can also use the idea above to obtain the volume formula for a ball of radius R.*

A ball of radius R can be described to be the region $x^2 + y^2 + z^2 \leq R^2$ on a Cartesian coordinate system. We can subdivide the ball into n horizontal pieces of equal height $\Delta z = \frac{2R}{n}$, and let $r_0 = -R, r_1 = -R + \frac{2R}{n}, ..., r_n = R$. When n is big, the volume of the i-th piece is close to a cylinder with radius $\sqrt{R^2 - r_i^2}$ and height Δz, which is

$$R_n = \sum_{i=1}^{n} \pi(\sqrt{R^2 - r_i^2})^2 \Delta z = \sum_{i=1}^{n} \pi(R^2 - r_i^2)\Delta z$$

We get the volume for the ball to be

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(R^2 - r_i^2)\Delta z = \int_{-R}^{R} \pi(R^2 - r^2)\, dx$$

An antiderivative of $1 - r^2$ is $R^2 r - \frac{r^3}{3}$, so

$$\begin{aligned}\int_{-R}^{R} \pi(R^2 - r^2)\, dx &= \pi \int_{-R}^{R} (R^2 - r^2)\, dx \\ &= \pi(R^2 r - \frac{r^3}{3})\Big|_{-R}^{R} \\ &= \frac{4}{3}\pi R^3\end{aligned}$$

Example 13. *In this example we will get the surface area of a sphere of radius R.*

Given a ball of radius R, there is another way to compute its volume: divide the interval $[0, R]$ into n subintervals of equal length Δr to decompose the ball into shells. If we denote the surface area of a sphere of radius r to be $S(x)$, then we see the volume of the ball can be approximated by

$$R_n = \sum_{i=1}^{n} f(r_i)\Delta r$$

Taking the limit as $n \to \infty$, we get the volume:

$$\int_0^R f(r)\,dr$$

By Part (i) of the theorem, we know $\frac{d}{dR}\int_0^R f(r)\,dr = f(R)$, and on the other hand, we have already computed in the previous example that $\int_0^R f(r)\,dr = \frac{4}{3}\pi R^3$, so

$$f(R) = \frac{d}{dR}(\frac{4}{3}\pi R^3) = 4\pi R^2$$